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Eigenpair Derivative with Respect to Boundary Shape

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This work focuses on how changes of boundary shapes affect eigenvalues and eigenfunctions in continuous systems. The governing equation, plus its boundary conditions for the eigenpair derivatives, is derived using the δ-function method, which can facilitate the differentiation of boundary conditions with respect to boundary shapes. Even though the eigenproblem equation with its corresponding boundary conditions is homogeneous, the governing equation with its boundary conditions for the eigenpair derivatives may be nonhomogeneous. A transformation method is proposed to transform the differential equation with nonhomogeneous boundary conditions into a new problem with homogeneous boundary conditions so that the eigenfunctions form a complete set for this new problem. The explicit results for the eigenpair derivatives are given, and an example is presented to illustrate the method and its validity.

I. Introduction

DESIGN derivatives of eigenpairs with respect to design variables are particularly useful in the design of structural vibrational systems and attract much interest in research fields. Although design derivatives of eigenpairs with respect to the changes in sizing variables (such as thickness of a plate) have been widely and deeply investigated, design derivatives of eigenpairs with respect to the changes in boundary shapes are less developed. Rousselet and Haug[1] formulated the dependence of eigenvalues on the shape of plates and plane elastic solids. Liu et al.[2] and Hu et al.[3] presented a method to formulate the derivatives of eigenpairs of a beam with respect to in-span support location. Wang[4] used the classical normal modal method to derive the formulas of eigenvalue derivatives with respect to in-span support location. Chuang and Hou[5] and Hou and Chuang[6] used the material derivative method to derive the formulas of eigenvalue derivatives with respect to in-span support locations. Liu and Mote[7] addressed the discrete approach for calculation of eigenvalue derivatives with respect to constraint locations using generalized Rayleigh principle. However, the derivation of eigenpair derivatives with respect to boundary shape appears to have not been brought to closure. In particular, the derivatives of eigenfunctions with respect to boundary shapes need to be further addressed. In this paper, a method is presented to derive explicit results for the eigenpair derivatives with respect to boundary shape variables. The δ-calculus method is used to facilitate the differentiation of the boundary conditions with respect to the boundary shapes. Even though the eigenproblem equation with its corresponding boundary conditions is homogeneous, the governing equation with its boundary conditions for the eigenpair derivatives may be nonhomogeneous. A transformation method is proposed to transform the differential equation with nonhomogeneous boundary conditions into a new problem with homogeneous boundary conditions so that the eigenfunctions form a complete set for this new problem. An example is presented to illustrate the method and its validity.

II. Technical Description

The closed, bounded, and regular domain of an elastic structure is denoted by $\bar{\Omega}$ for which the interior is Ω and boundary is Γ. A typical point in Ω is denoted by x. The class of eigenvalue problems under discussion is defined through the operator equation

$$L[w_i] - \lambda_i M w_i = 0 \qquad x \in \Omega \tag{1}$$

$$B_r[w_i]\delta(x - x_B) = 0 \qquad x_B \in \Gamma \qquad r = 1, 2, \ldots, p \tag{2}$$

where λ_i is the ith eigenvalue and $w_i(x)$ is the eigenfunction corresponding to λ_i, L is linear homogeneous differential operators of order $2p$, operator M is only a function of the spatial variables, B_r is a linear homogeneous differential operator involving derivatives normal to the boundary and along the boundary through order $2p - 1$, and $\delta(x)$ is a Dirac delta function. Here we assume that $\lambda_1 < \lambda_2 < \lambda_3 \ldots$.

For simplicity, we assume that the operators B_r depend on the shape of the domain and the operators L and M are independent of the shape of the domain.

Changes in the boundary shape Γ are defined by a vector field $V(x)$ (see Fig. 1). By definition, a typical point $x_\tau \in \Omega_\tau$ is given by

$$x_\tau = x + \tau V(x) \qquad x \in \Omega$$

where τ is a scalar.

The eigenproblem with perturbed boundary shape can be written as

$$L[w_{i\tau}] - \lambda_{i\tau} M w_{i\tau} = 0 \qquad x_\tau \in \Omega_\tau \tag{3}$$

$$B_{r\tau}[w_{i\tau}]\delta(x - x_{B\tau}) = 0 \qquad x_{B\tau} \in \Gamma_\tau \qquad r = 1, 2, \ldots, p \tag{4}$$

where $w_{i\tau}$ and $\lambda_{i\tau}$ are the ith eigenpair associated with the perturbed boundary shape and $B_{r\tau}$ is the perturbed boundary operator.

The eigenpair derivatives with respect to boundary shapes are defined by

$$\dot{w}_i(x) \equiv \lim_{\tau \to 0} \frac{w_{i\tau}(x) - w_i(x)}{\tau} \tag{5}$$

$$\dot{\lambda}_i \equiv \lim_{\tau \to 0} \frac{\lambda_{i\tau} - \lambda_i}{\tau} \tag{6}$$

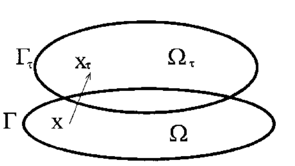

Fig. 1 Boundary shape change, $x_\tau = x + \tau V(x)$.

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III. Governing Equation Plus Boundary Conditions for $\dot{\lambda}_i$ and $\dot{w}_i(x)$

The governing equation with boundary conditions for $\dot{w}_i$ and $\dot{\lambda}_i$ can be obtained by differentiating Eqs. (3) and (4) with respect to τ at $\tau = 0$, respectively.

Differentiation of Eq. (3) with respect to τ at $\tau = 0$ gives

$$L[\dot{w}_i] - \lambda_i M\dot{w}_i = \dot{\lambda}_i M w_i \qquad x \in \Omega \tag{7}$$

Differentiation of Eq. (4) with respect to τ at $\tau = 0$, using the chain rules of differentiation, yields

$$B_r[\dot{w}_i]\delta(x - x_B) + \dot{B}_r[w_i]\delta(x - x_B) + B_r[w_i]\left.\frac{\partial \delta(x - x_{B\tau})}{\partial \tau}\right|_{\tau=0} = 0 \qquad x_B \in \Gamma \qquad r = 1, 2, \ldots, p \tag{8}$$

where $\dot{B}_r$ is defined by

$$\dot{B}_r[w] \equiv \lim_{\tau \to 0} \frac{B_{r\tau}[w] - B_r[w]}{\tau} \tag{9}$$

Using the formula[8] of δ-calculus (see the Appendix), Eq. (8) can be rewritten as

$$B_r[\dot{w}_i]\delta(x - x_B) = -\dot{B}_r[w_i]\delta(x - x_B) - \{\mathrm{grad}(B_r[w_i]) \cdot V(x)\}\delta(x - x_B) \qquad x_B \in \Gamma \qquad r = 1, 2, \ldots, p \tag{10}$$

where the dot between $\mathrm{grad}(B_r[w_i])$ and $V(x)$ denotes the scalar product of the two vectors. Equations (7) and (10) form the governing equation with boundary conditions for $\dot{w}_i$ and $\dot{\lambda}_i$.

IV. Transformation of Equations (7) and (10) with Nonhomogeneous Boundary Conditions

It is desirable to obtain the solution to Eqs. (7) and (10) using the available eigenpairs of Eqs. (1) and (2) instead of solving Eqs. (7) and (10) directly. Let us try the modal summation method because it is a typical method to be used for such purpose. However, the modal summation method cannot be directly applied to solution of Eqs. (7) and (10) in their current form because of the nonhomogeneous boundary conditions in Eq. (10). To deal with this problem, we need to transform the operator of Eqs. (7) and (10) with nonhomogeneous boundary conditions to ones with homogeneous boundary conditions.

Suppose that we can find a function $f(x)$ that satisfies the specified boundary conditions in Eq. (10),

$$B_r[f]\delta(x - x_B) = -\dot{B}_r[w_i]\delta(x - x_B) - \{\mathrm{grad}(B_r[w_i]) \cdot V(x)\}\delta(x - x_B) \qquad x_B \in \Gamma \qquad r = 1, 2, \ldots, p$$

and is sufficiently smooth so that $(L[f] - \lambda_i M f) \in L_2(\Omega)$. Then $z(x) = \dot{w}_i(x) - f(x)$ satisfies the equation

$$L[z] - \lambda_i M z = \dot{\lambda}_i M w_i + \lambda_i M f - L[f] \qquad x \in \Omega \tag{11}$$

together with homogeneous boundary conditions

$$B_r[z]\delta(x - x_B) = 0 \qquad x_B \in \Gamma \qquad r = 1, 2, \ldots, p \tag{12}$$

Thus we are able to transform the operator of Eqs. (7) and (10) with nonhomogeneous boundary conditions to ones with homogeneous boundary conditions. Once $z(x)$ is determined from Eqs. (11) and (12), the solution to Eqs. (7) and (10) is given by

$$\dot{w}_i(x) = z(x) + f(x) \tag{13}$$

V. Determination of $f(x)$

Consider the problem of determining $f(x)$, which is only required to satisfy the specified boundary conditions,

$$B_r[f]\delta(x - x_B) = -\dot{B}_r[w_i]\delta(x - x_B) - \{\mathrm{grad}(B_r[w_i]) \cdot V(x)\}\delta(x - x_B) \qquad x_B \in \Gamma \qquad r = 1, 2, \ldots, p$$

In fact, there are many $f(x)$ that satisfy the preceding boundary conditions. Since we have some room to select $f(x)$ from its kind, it is desirable to select such $f(x)$ that it contributes significantly to $\dot{w}_i$ and takes fewer efforts to be determined. In this case, it is possible to give an accurate $\dot{w}_i$ even though smaller numbers of the first eigenfunctions are used in modal summation method for obtaining $z(x)$. Liu et al.[9] and Hu[10] have shown that the accuracy for the modal summation method can be dramatically improved if a static response is added. Therefore, the solution to the following static response problem to initial boundary deformations and/or forces is specified as $f(x)$:

$$L[f] = 0 \qquad x \in \Omega \tag{14}$$

with boundary conditions

$$B_r[f]\delta(x - x_B) = -\dot{B}_r[w_i]\delta(x - x_B) - \{\mathrm{grad}(B_r[w_i]) \cdot V(x)\}\delta(x - x_B) \qquad x_B \in \Gamma \qquad r = 1, 2, \ldots, p \tag{15}$$

It can be seen that it is not difficult to obtain $f(x)$ from Eqs. (14) and (15), which is a static response problem to the specified initial boundary deformations or/and forces.

VI. Determination of $\dot{\lambda}_i$ and $z(x)$ from Eqs. (11) and (12)

After $f(x)$ is determined, $\dot{\lambda}_i$ and $z(x)$ can be obtained from Eqs. (11) and (12). Since the boundary conditions for $z(x)$ in Eqs. (11) and (12) are homogeneous, being compatible to Eqs. (1) and (2), the eigenfunctions $w_j(x)$ form a complete set for $z(x)$. Therefore, $z(x)$ can be expressed using the modal summation

$$z(x) = \sum^{\infty} c_j w_j(x) \tag{16}$$

where c_j is a parameter to be determined.

A. Domain Integration for $\dot{\lambda}_i$, c_j, and c_i

For simplicity, we assume that λ_i is distinct. In case that λ_i is repeated or closely spaced, $\dot{\lambda}_i$ should be determined from a subeigenproblem.[10–12] Introduction of Eq. (16) into Eq. (11), multiplication of both sides of Eq. (11) with $w_i(x)$, and then integration of the resultant equation in Ω yield

$$\dot{\lambda}_i = -\int_{\Omega} (\lambda_i M w_i) f(x)\, \mathrm{d}\Omega \tag{17}$$

where the modal normalization,

$$\int_{\Omega} M w_i^2\, \mathrm{d}\Omega = 1$$

is imposed.

Introduction of Eq. (16) into Eq. (11), multiplication of Eq. (11) with $w_j(x)$, and then integration of the resultant equation in Ω, after orthogonality of eigenfunctions is applied give

$$c_j = \frac{\lambda_i}{\lambda_j(\lambda_j - \lambda_i)} \int_{\Omega} (\lambda_j M w_j) f(x)\, \mathrm{d}\Omega \qquad j \neq i \tag{18}$$

To determine c_i we need to use the normalization equation of $w_{i\tau}(x)$,

$$\int_{\Omega_\tau} M w_{i\tau}^2\, \mathrm{d}\Omega_\tau = 1$$

Differentiation of the preceding expression with respect to τ at $\tau = 0$ gives

$$\left[\frac{\partial}{\partial\tau}\int_{\Omega_\tau} M w_{i\tau}^2 \,\mathrm{d}\Omega_\tau\right]\Bigg|_{\tau=0} = 0 \tag{19}$$

Based on the formula provided in Ref. 11 (p. 197, Lemma 3.2.1), Eq. (19) can be further rewritten as

$$\int_\Omega 2Mw_i\dot{w}_i\,\mathrm{d}\Omega + \int_\Gamma Mw_i^2[V(x)\cdot n]\,\mathrm{d}\Gamma = 0 \tag{20}$$

where n denotes the unit vector normal to the boundary Γ and out of the domain.

Introduction of Eqs. (13) and (16) into Eq. (20) yields

$$c_i = -\frac{1}{\lambda_i}\int_\Omega(\lambda_i M w_i) f(x)\,\mathrm{d}\Omega - \frac{1}{2\lambda_i}\int_\Gamma(\lambda_i M w_i^2)[V(x)\cdot n]\,\mathrm{d}\Gamma \tag{21}$$

Equations (17), (18), and (21) are the domain integration expressions for $\dot{\lambda}_i$, c_j, and c_i, respectively.

B. Boundary Integration for $\dot{\lambda}_i$ and c_j

After substitution of $L[w_j]$ for $\lambda_j M w_j$ in Eqs. (17), (18), and (21), we have thus in place of Eqs. (17), (18), and (21)

$$\dot{\lambda}_i = -\int_\Omega L[w_i] f(x)\,\mathrm{d}\Omega \tag{22}$$

$$c_j = \frac{\lambda_i}{\lambda_j(\lambda_j - \lambda_i)}\int_\Omega L[w_j] f(x)\,\mathrm{d}\Omega \qquad j \neq i \tag{23}$$

$$c_i = -\frac{1}{\lambda_i}\int_\Omega L[w_i] f(x)\,\mathrm{d}\Omega - \frac{1}{2\lambda_i}\int_\Gamma(\lambda_i M w_i^2)[V(x)\cdot n]\,\mathrm{d}\Gamma \tag{24}$$

On many occasions it is possible to perform an integration by parts on

$$\int_\Omega L[w_j] f(x)\,\mathrm{d}\Omega$$

and replace it by an alternative statement of the form

$$\int_\Omega L[w_j] f(x)\,\mathrm{d}\Omega = \int_\Gamma E[w_j] F[f]\,\mathrm{d}\Gamma \tag{25}$$

In this the operators E and F contain lower-order derivatives than those occurring in operator L. As can be seen from Eq. (15), the boundary values of $f(x)$ can be linked explicitly to the boundary values of $w_i(x)$, and thus one has

$$F[f]\delta(x - x_B) = C[w_i]\delta(x - x_B) \tag{26}$$

where C is boundary operator.

After introduction of Eqs. (25) and (26) into Eqs. (22–24), the domain integration expressions Eqs. (17), (18), and (21) can be transformed into their boundary integration expressions

$$\dot{\lambda}_i = -\int_\Gamma E[w_i] C[w_i]\,\mathrm{d}\Gamma \tag{27}$$

$$c_j = \frac{\lambda_i}{\lambda_j(\lambda_j - \lambda_i)}\int_\Gamma E[w_j] C[w_i]\,\mathrm{d}\Gamma \qquad j \neq i \tag{28}$$

$$c_i = -\frac{1}{\lambda_i}\int_\Gamma E[w_i] C[w_i]\,\mathrm{d}\Gamma - \frac{1}{2\lambda_i}\int_\Gamma(\lambda_i M w_i^2)[V(x)\cdot n]\,\mathrm{d}\Gamma \tag{29}$$

Therefore, $\dot{\lambda}_i$, c_j, and c_i can be obtained without $f(x)$ involved when the boundary integration Eqs. (27–29) are applied.

C. Determination of $\dot{w}_i(x)$

After obtaining $f(x)$, c_j, and c_i, $\dot{w}_i$ can be determined from Eqs. (13) and (16), i.e.,

$$\dot{w}_i = f(x) + \sum_{j=1}^{\infty} c_j w_j(x) \approx f(x) + \sum_{j=1}^{N} c_j w_j(x) \tag{30}$$

When Eq. (30) is used in practical computations, only the first few available eigenfunctions are included in the series with unavailable higher eigenfunctions truncated. Therefore, the accuracy of Eq. (30) is an important issue. As can be seen from Eq. (28), the denominator of c_j is $\lambda_j(\lambda_j - \lambda_i)$. This means that the convergent rate of c_j is in λ_j^{-2}.

VII. Example

Consider the flexible vibration of the beam shown in Fig. 2. The corresponding eigenproblem is

$$L[w_i] - \lambda_i M w_i = 0 \qquad 0 < x < l$$

together with boundary conditions

$$B_1 w_i(x)\delta(x - x_B) = 0 \qquad x_B = 0, l$$

$$B_2 w_i(x)\delta(x - x_B) = 0 \qquad x_B = 0, l$$

where $L \equiv EJ(\partial^4/\partial x^4)$, $M \equiv \rho$, $B_1 \equiv 1$, and $B_2 \equiv EJ(\partial^2/\partial x^2)$. Its ith eigenfunction and ith eigenvalue are

$$w_i(x) = \sqrt{(2/\rho l)}\sin(i\pi x/l) \qquad \lambda_i = (i\pi/l)^4(EJ/\rho)$$

Suppose that the shape design velocity field is given by $V(x) = x/l$. Then the length of the beam with perturbed domain equals to $l + \tau$. Thus the ith eigenfunction and the ith eigenvalue of the beam with the perturbed domain are

$$w_{i\tau}(x) = \sqrt{\frac{2}{\rho(l+\tau)}}\sin\frac{i\pi x}{l+\tau} \qquad \lambda_i = \left(\frac{i\pi}{l+\tau}\right)^4\left(\frac{EJ}{\rho}\right)$$

A. Analytical Expressions for $\dot{\lambda}_i$ and $\dot{w}_i$

Differentiation of $\lambda_{i\tau}$ and $w_{i\tau}(x)$ given earlier with respect to τ gives the analytical expressions for $\dot{w}_i$ and $\dot{\lambda}_i$,

$$\dot{w}_i(x) = -\frac{1}{l\sqrt{2\rho l}}\sin\frac{i\pi x}{l} - \frac{i\pi x}{l^2}\sqrt{\frac{2}{\rho l}}\cos\frac{i\pi x}{l}$$

$$\dot{\lambda}_{i\tau} = -4(i\pi/l)^4(EJ/\rho l)$$

B. Determination of $\dot{\lambda}_i$ and $\dot{w}_i$ Using the Present Method

From Eqs. (14) and (15), one obtains the governing equation for $f(x)$

$$EJ\frac{\partial^4 f(x)}{\partial x^4} = 0 \qquad 0 < x < l$$

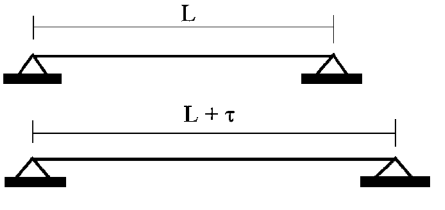

Fig. 2 Perturbed boundary beam with $V(x) = x/L$.

with boundary conditions

$$f(x)\delta(x-x_B) = -\frac{\partial w_i}{\partial x}V(x)\delta(x-x_B) \qquad x_B = 0, l$$

$$EJ\frac{\partial^2 f}{\partial x^2}\delta(x-x_B) = -EJ\frac{\partial^3 w_i}{\partial x^3}V(x)\delta(x-x_B) \qquad x_B = 0, 1$$

which is just a static response problem to the initial boundary deformations and/or forces.

The solution to this problem is easily obtained:

$$f(x) = \left[\frac{l}{6}\frac{\partial^3 w_i(x)}{\partial x^3}\bigg|_{x=l} - \frac{1}{l}\frac{\partial w_i(x)}{\partial x}\bigg|_{x=l}\right]x - \left[\frac{1}{6l}\frac{\partial^3 w_i(x)}{\partial x^3}\bigg|_{x=l}\right]x^3$$

Thus, $\dot{\lambda}_i$ can be determined using the direct domain integration Eq. (17):

$$\dot{\lambda}_i = -\int_0^l \lambda_i \rho w_i(x) f(x)\,\mathrm{d}x = 2\left(EJ\frac{\partial^3 w_i}{\partial x^3}\bigg|_{x=l}\right)\frac{\partial w_i}{\partial x}\bigg|_{x=l} = -4\left(\frac{i\pi}{l}\right)^4\frac{EJ}{\rho l}$$

As an alternative, $\dot{\lambda}_i$ can also be determined from the boundary integration Eq. (22):

$$\begin{aligned}\dot{\lambda}_i &= -\int_0^l EJ\frac{\partial^4 w_i(x)}{\partial x^4}f(x)\,\mathrm{d}x \\ &= -\left[EJ\frac{\partial^3 w_i(x)}{\partial x^3}f(x) - EJ\frac{\partial^2 w_i(x)}{\partial x^2}\frac{\partial f}{\partial x} + EJ\frac{\partial w_i(x)}{\partial x}\frac{\partial^2 f}{\partial x^2} - EJw_i\frac{\partial^3 f}{\partial x^3}\right]\bigg|_{x=l} \\ &\quad + \left[EJ\frac{\partial^3 w_i(x)}{\partial x^3}f(x) - EJ\frac{\partial^2 w_i(x)}{\partial x^2}\frac{\partial f}{\partial x} + EJ\frac{\partial w_i(x)}{\partial x}\frac{\partial^2 f}{\partial x^2} - EJw_i\frac{\partial^3 f}{\partial x^3}\right]\bigg|_{x=0} \\ &= 2\left(EJ\frac{\partial^3 w_i}{\partial x^3}\bigg|_{x=l}\right)\frac{\partial w_i}{\partial x}\bigg|_{x=l} \\ &= -4\left(\frac{i\pi}{l}\right)^4\frac{EJ}{\rho l}\end{aligned}$$

In the same way as determination of $\dot{\lambda}_i$, one can obtain c_j and c_i using the domain integration Eqs. (18) and (21) or the boundary integration Eqs. (28) and (29):

$$\begin{aligned}c_j &= -\frac{\lambda_i}{\lambda_j(\lambda_j - \lambda_i)}\int_0^l \lambda_j \rho w_j f(x)\,\mathrm{d}x \\ &= -\frac{\lambda_i}{\lambda_j(\lambda_j - \lambda_i)}\left\{\left[EJ\frac{\partial^3 w_i(x)}{\partial x^3}\right]\frac{\partial w_j(x)}{\partial x} + \left[EJ\frac{\partial^3 w_j(x)}{\partial x^3}\right]\frac{\partial w_i(x)}{\partial x}\right\}\bigg|_{x=l} \qquad j \neq i\end{aligned}$$

$$\begin{aligned}c_i &= -\frac{1}{\lambda_i}\int_0^l \lambda_i \rho w_i f(x)\,\mathrm{d}x - \frac{1}{2\lambda_i}\left\{\left[\lambda_i \rho w_i^2 V(x)\right]\Big|_{x=l} - \left[\lambda_i \rho w_i^2 V(x)\right]\Big|_{x=0}\right\} \\ &= 2\frac{1}{\lambda_i}\left\{\left[EJ\frac{\partial^3 w_i(x)}{\partial x^3}\right]\frac{\partial w_i(x)}{\partial x}\right\}\bigg|_{x=l}\end{aligned}$$

Table 1 Comparison of error for eigenfunction derivative $\dot{w}_1$

Number of the first eigenfunctions used in Eq. (30)	1	2	3	4
Error, %, with $f(x)$ included in Eq. (30)	0.0174	0.0022	0.0001	0.0000
Error, %, without $f(x)$ included in Eq. (30)	494.6	489.3	489.0	488.9

Table 2 Comparison of error for eigenfunction derivative $\dot{w}_2$

Number of the first eigenfunctions used in Eq. (30)	2	3	4	5
Error, %, with $f(x)$ included in Eq. (30)	1.5058	0.0495	0.0045	0.0007
Error, %, without $f(x)$ included in Eq. (30)	2780	2767	2766	2766

Table 3 Comparison of error for eigenfunction derivative $\dot{w}_3$

Number of the first eigenfunctions used in Eq. (30)	3	4	5	6
Error, %, with $f(x)$ included in Eq. (30)	3.6163	0.1966	0.0250	0.0048
Error, %, without $f(x)$ included in Eq. (30)	10,858	10,840	10,838	10,837

Table 4 Comparison of error for eigenfunction derivative $\dot{w}_4$

Number of the first eigenfunctions used in Eq. (30)	4	5	6	7
Error, %, with $f(x)$ included in Eq. (30)	5.8718	0.4380	0.0696	0.0159
Error, %, without $f(x)$ included in Eq. (30)	31,164	31,143	31,140	31,139

Table 5 Analysis of eigenfunction derivative $\dot{w}_1$

	x					
Coordinate of x	0	0.21	0.41	0.61	0.81	1
$f(x)$	0.00	2.29	4.23	5.47	5.65	4.44
$C_1 W_1(x)$	0.00	−3.32	−5.38	−5.38	−3.32	0.00
$\dot{w}_1 \approx f(x) + c_1 w(x)$	0.00	−1.03	−1.15	0.09	2.33	4.44
$\dot{w}_1(x)$ (exact)	0.00	−1.13	−1.22	0.15	2.45	4.44

As can be seen from the preceding development, the explicit expression for $\dot{\lambda}_i$ matches exactly its analytical expression.

C. Numerical Comparison

To illustrate numerically the convergent rate of Eq. (30) and the contribution of $f(x)$ to $\dot{w}_i$, the following numerical comparisons are listed in Tables 1–5.

In this numerical example, the parameters of the beam shown in Fig. 2 are given by $EJ = 1.0 \times 10^{-4}$ Nm^2, $l = 1.0$ m, and ρ (mass per length) $= 1.0$ kg/m. For comparison, the error is defined as

$$\text{error} \equiv \frac{\sum_{j=0}^{n}\left\{\dot{w}_i[(j/n)l] - \dot{w}_i^a[(j/n)l]\right\}^2}{\sum_{j=1}^{n}\left\{\dot{w}_i[(j/n)l]\right\}^2}$$

where $\dot{w}_i^a(x)$ is the approximate to $\dot{w}_i(x)$.

The numerical values in Tables 1–5 confirm the following:

1) The term $f(x)$ contributes significantly to $\dot{w}_i$. If $f(x)$ is not included in the series of Eq. (30), the final result will be misleading no matter how many terms of eigenfunctions are included in that series.

2) The convergent rate is fast in terms of the number of first eigenfunctions used in Eq. (30). When the number of the first

eigenfunctions used for summation is about $i + 2$, the error can be reduced to less than 1.0%.

VIII. Conclusion

For distributed parameter systems, exact design derivatives of eigenvalues and eigenfunctions with respect to boundary shapes are derived in this paper. A beam example is given to illustrate the method and its validity. The results show that the set of eigenfunctions is not complete for $\dot{w}_i(x)$ in the sense of modal summation method for calculation of $\dot{w}_i(x)$ due to the nonhomogeneous boundary conditions of $\dot{w}_i(x)$. However, the set of the eigenfunctions $w_j(x)$ can be complete for $z(x)$ after a transformation $\dot{w}_i(x) = z(x) + f(x)$ is introduced. Therefore, $f(x)$, which is easy to evaluate by solving a static response problem to the specified boundary deformations and/or forces, must not be ignored when calculating $\dot{w}_i(x)$ using Eq. (30). The results also show that the denominator of c_j is proportial to $\lambda_j(\lambda_j - \lambda_i)$. This means that the convergent rate in the series of Eq. (30) is λ_j^{-2}. Such satisfactory convergent property is attributed to $f(x)$.

Appendix: δ-Function Properties

For the convenience of readers, we have listed the definitions and major properties of δ-function used in this paper, which can be found in the textbooks of mathematical physics.

Definitions:

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\,\mathrm{d}x \equiv f(0) \tag{A1}$$

$$\int_{-\infty}^{+\infty} f(x)\frac{\mathrm{d}\delta(x)}{\mathrm{d}x}\,\mathrm{d}x \equiv -\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=0} \tag{A2}$$

Properties:

$$\int_{-\infty}^{+\infty} f(x-a)\delta(x)\,\mathrm{d}x = f(a) \tag{A3}$$

$$\int_{-\infty}^{+\infty} f(x)\frac{\mathrm{d}\delta(x-a)}{\mathrm{d}a}\,\mathrm{d}x = \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=a} \tag{A4}$$

In a similar way, one can extend the δ-function mentioned earlier to multiple dimensions:

$$\int_{\Omega} f(x)\delta(x - x_{B\tau})\,\mathrm{d}\Omega \equiv f(x_{B\tau}) \tag{A5}$$

where x and $x_{B\tau}$ are vectors and $x_{B\tau} = x + \tau V(x)$.

Differentiation of the preceding equation with respect to τ at $\tau = 0$ gives

$$\int_{\Omega} f(x)\left.\frac{\mathrm{d}\delta(x - x_{B\tau})}{\mathrm{d}\tau}\right|_{\tau=0}\mathrm{d}\Omega = \{\mathrm{grad}[f(x)]\}\cdot V(x)\delta(x - x_B) \tag{A6}$$

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References

[1] Rousselet, B., and Haug, E. J., "Design Sensitivity Analysis in Structural Mechanics. 3. Effects of Shape Variation," *Journal of Structural Mechanics*, Vol. 10, No. 3, 1982–1983, pp. 273–310.

[2] Liu, Z. S., Hu, H. C., and Wang, D. J., "New Method for Deriving Rate Eigenvalue with Respect to Support Location," *AIAA Journal*, Vol. 34, No. 4, 1996, pp. 864–866.

[3] Hu, H. C., Liu, Z. S., and Wang, D. J., "The Effects of Changes in Constraint Locations on Vibration Modes," *Acta Mechanica Sinica*, Vol. 28, No. 1, 1996, pp. 23–32.

[4] Wang, B. P., "Eigenvalue Sensitivity with Respect to Location of Internal Stiffness and Mass Attachments," *AIAA Journal*, Vol. 31, No. 4, 1993, pp. 791–793.

[5] Chuang, C. H., and Hou, G. J. W., "Eigenvalue Sensitivity Analysis of Planar Frames with Variable Joint and Support Locations," *AIAA Journal*, Vol. 30, No. 8, 1992, pp. 2138–2147.

[6] Hou, G. J. W., and Chuang, C. H., "Eigenvector Sensitivity Analysis of Continuous Beams with Variable Support Locations," *Fourth International Conference in Recent Advances in Structural Dynamics* (London), edited by M. Petyt, H. F. Wolfe, and C. Mei, 1991, pp. 313–322.

[7] Liu, Z. S., and Mote, C. D., "A New Discrete Approach to Eigenvalue Sensitivity with Respect to Constraint Locations," *Journal of Sound and Vibration* (submitted for publication).

[8] Butkov, E., *Mathematical Physics*, Addison–Wesley, Reading, MA, 1968, pp. 221–235.

[9] Liu, Z. S., Chen, S. H., Yu, M., and Zhao, Y. Q., "Contribution of the Truncated Modes to Eigenvector Derivatives," *AIAA Journal*, Vol. 32, No. 7, 1994, pp. 1551–1553.

[10] Hu, H. C., *Vibration Theory of Structures with MDOF*, Science Press, Beijing, PRC, 1987.

[11] Haug, E. J., Choi, K. K., and Komkov, V., *Design Sensitivity Analysis of Structural Systems*, Academic, New York, 1986, p. 197.

[12] Chen, S. H., Liu, Z. S., Shao, C. S., and Zhao, Y. Q., "Perturbation Analysis of Vibration Modes with Close Frequencies," *Communications in Numerical Methods in Engineering*, Vol. 9, No. 5, 1993, pp. 427–438.